Perturbative estimates on the transport cross section in quantum scattering by hard obstacles

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Abstract

The quantum scattering by smooth bodies is considered for small and large values of kd, with k the wavenumber and d the scale of the body. In both regimes, we prove that the forward scattering exceeds the backscattering. For high k, we need to assume that the body is strictly convex.

Key words: quantum scattering, transport cross section

1 Introduction

1.1 Quantum scattering

We briefly present in physical language the quantum scattering problem for hard objects in three dimensions. Fix a z-axis in $\mathbb{R}^3$ and denote the unit vector along that axis as $\mathbf{e} = (0, 0, 1) \in \mathbb{R}^3$. Let a body be given as a compact subset $\Omega \subset \mathbb{R}^3$ and consider a flow of free quantum particles with wave vector $\mathbf{k} = k\mathbf{e}$, incident on Ω. The body is modeled by a hardcore potential V_Ω,

$$V_\Omega(\mathbf{r}) = \begin{cases} 0 & \text{if } \mathbf{r} \notin \Omega, \\ +\infty & \text{if } \mathbf{r} \in \Omega. \end{cases} \tag{1.1}$$

Basic scattering theory [5] teaches us that far from the scatterer (in the limit $\mathbf{r} \uparrow \infty$), the wave function $\Psi(\mathbf{r})$ is obtained by adding an outgoing spherical wave $\frac{f(\mathbf{q})}{r}\mathrm{e}^{\mathrm{i}kr}$ to the incoming plane wave $\mathrm{e}^{\mathrm{i}kz}$.

$$\Psi(\mathbf{r}) \approx \mathrm{e}^{\mathrm{i}kz} + \frac{f(\mathbf{q})}{r}\mathrm{e}^{\mathrm{i}kr}, \qquad \mathbf{r} \in \mathbb{R}^3 \setminus \Omega,\ \mathbf{q} \in S^2 \tag{1.2}$$

where S^2 is the unit sphere: $\mathbf{q} \in S^2 \Leftrightarrow \mathbf{q} \cdot \mathbf{q} = 1$, $r := |\mathbf{r}|, k = |\mathbf{k}|$ and $z = \mathbf{r} \cdot \mathbf{e}$. This notation will be used throughout the paper. The function $f(\mathbf{q})$ goes under the name of *scattering amplitude*, it describes the form of the outgoing spherical wave. The *scattering amplitude*

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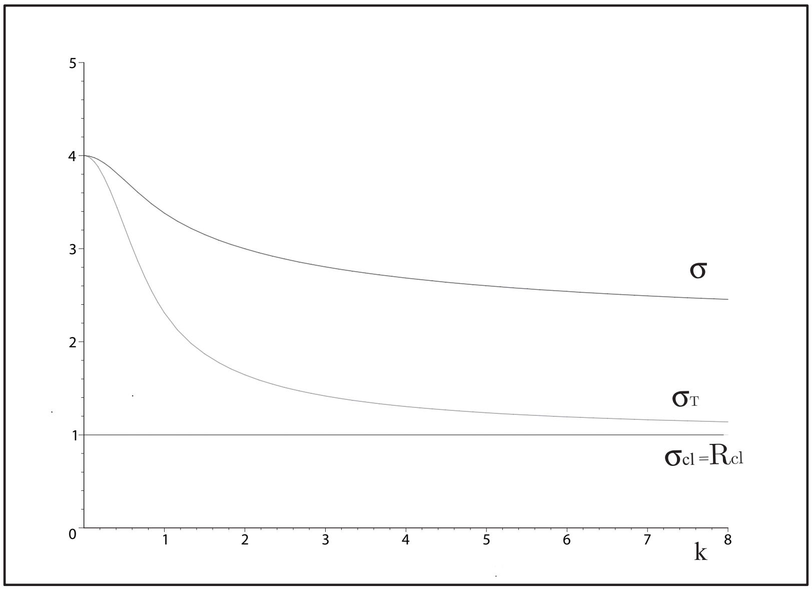

Figure 1: Transport cross section σ_T and classical resistance R_{cl}; total cross section σ and classical total cross section σ_{cl} for the hard sphere with radius $r = \pi^{-1/2}$.

depends on kd where d is the typical scale of Ω. For simplicity we keep Ω (and hence d) fixed and we vary k. The intensity of the scattered wave is given by the *total cross section*

$$\sigma = \int_{S^2} \mathrm{d}\mathbf{q}\, |f(\mathbf{q})|^2 \tag{1.3}$$

where $\mathrm{d}\mathbf{q}$ is the uniform measure on the sphere. We also consider the *momentum transfer cross section*, or *transport cross section* σ_T,

$$\sigma_T = \frac{1}{k} \int_{S^2} \mathrm{d}\mathbf{q}\, \mathbf{k} \cdot (\mathbf{e} - \mathbf{q}) |f(\mathbf{q})|^2 \tag{1.4}$$

Both σ and σ_T have the dimension of an *area*, justifying the name *cross section*. They can be computed explicitly for the sphere [1], see Fig. 1.1. We see that for all positive $k > 0$,

$$\sigma_T < \sigma \tag{1.5}$$

By some rewriting,

$$\begin{aligned} \sigma_T - \sigma &= \int_{S^2} \mathrm{d}\mathbf{q}\, \mathbf{e} \cdot (\mathbf{e} - \mathbf{q}) |f(\mathbf{q})|^2 - \int_{S^2} \mathrm{d}\mathbf{q}\, |f(\mathbf{q})|^2 && (1.6) \\ &= -\int_{S^2} \mathrm{d}\mathbf{q}\, \cos\theta |f(\mathbf{q})|^2 && (1.7) \end{aligned}$$

where θ denotes the angle between $\mathbf{q}$ and $\mathbf{e}$. We see that the inequality (1.5) means that the forward scattering is greater than the backscattering.

It is a well-known physical fact that at $k = 0$, the scattering is isotropic. Indeed, if we write $C(\Omega)$ for the capacity of Ω (defined further in (2.2)), then

$$f(\mathbf{q}) = -C + \mathcal{O}(k) \tag{1.8}$$

This was established rigourously in [3]. An obvious consequence is that, in lowest order in k, the *momentum transfer cross section* coincides with the *total cross section*,

$$\sigma_T + \mathcal{O}(k) = \sigma + \mathcal{O}(k) = 4\pi C^2. \tag{1.9}$$

Apart from this obvious fact, we know of no place in the literature where the relation between σ and σ_T is examined (quite in contrast to the classical case, see Section 1.2). More generally, we are not aware of any qualitative results on the *scattering amplitude* for small but nonzero $k > 0$, other than the optical theorem

$$\frac{4\pi}{k}\Im f(\mathbf{e}) = \sigma$$

A natural question seems to be how general the inequality (1.5) is. Remark that the optical theorem does not answer this question, although it does say that the forward scattering cannot vanish completely.

Our first result, Theorem 2.1, establishes the inequality (1.5) perturbatively up to order k^3 for a general class of bodies. Our second result, Theorem 2.2, establishes the inequality (1.5) for large k.

1.2 Classical analogue

We briefly construct the *classical scattering amplitude* f_{cl} associated to a body Ω.

Consider a flow of *classical* particles with momentum $\mathbf{k}$, incident on Ω. The particles will move freely, then undergo several[1] elastic collisions with Ω and finally move freely again with momentum $\mathbf{k}^+(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^2$ marks their initial coordinates in $\mathbf{e}^\perp$, the plane perpendicular to $\mathbf{e}$. Since the collisions are assumed elastic, we have $|\mathbf{k}^+| = k$.
Let $\mathcal{I} \subset \mathbb{R}^2$ be the shadow associated to Ω, i.e.

$$\mathbf{x} \in \mathcal{I} \Leftrightarrow \exists z \in \mathbb{R} : (\mathbf{x}, z) \in \Omega \tag{1.10}$$

Let F be the map from $\mathcal{I}$ to the sphere S^2 such that $F(\mathbf{x}) = \frac{\mathbf{k}^+(\mathbf{x})}{k}$. Assuming strict convexity of Ω, the inverse F^{-1} exists (possibly up to a set of measure zero). We define the *classical scattering amplitude* as

$$f_{\mathrm{cl}}(\mathbf{q}) = |J(F^{-1})(\mathbf{q})|^{1/2}, \qquad \mathbf{q} \in S^2 \tag{1.11}$$

where $J(F^{-1})$ is the Jacobian determinant of the map F^{-1}. Now one can define the *classical resistance* R_{cl} and the *classical cross section* σ_{cl} in analogy to (1.4) as

$$\sigma_{\mathrm{cl}} = \int_{S^2} \mathrm{d}\mathbf{q}\, |f_{\mathrm{cl}}(\mathbf{q})|^2 \qquad R_{\mathrm{cl}} = \int_{S^2} \mathrm{d}\mathbf{q}\, \cos\theta |f_{\mathrm{cl}}(\mathbf{q})|^2 \tag{1.12}$$

[1] For nonconvex bodies, it can happen that incoming particles undergo an infinite number of collisions. Excluding this possibility requires an additional assumption.

which is equivalent to the more straightforward definitions

$$\sigma_{\text{cl}} = \int_{\mathcal{I}} \mathrm{d}\mathbf{x} = |\mathcal{I}|, \qquad R_{\text{cl}} = \frac{1}{k} \int_{\mathcal{I}} \mathrm{d}\mathbf{x}\, \mathbf{e} \cdot (\mathbf{k} - \mathbf{k}^{+}(\mathbf{x})) \tag{1.13}$$

(In fact, the function f_{cl} can be infinite on a set of measure zero, but it remains integrable. This follows e.g. by rewriting it as (1.13).)

At this point one can ask some interesting questions: Already Newton [2] posed and solved the problem of minimizing R_{cl} in the class of axially symmetric convex bodies inscribed in a fixed cylinder. Recently, this problem has received renewed attention, see e.g. [4]. The quantum analogue of this problem; minimizing σ_T while keeping σ fixed, seems by far out of reach.

2 Results

Assume for simplicity that Ω is a compact body with smooth surface, i.e. it is in the class C^∞. We rewrite (1.2) as a bounday value problem. Let u be a function on $\mathbb{R}^3 \setminus \Omega$, satisfying

1. The Helmholtz equation $(\triangle + k^2)u = 0$
2. The boundary condition $u(\mathbf{x}) = -\mathrm{e}^{\mathrm{i}kz}$ for $\mathbf{x} \in \delta\Omega$
3. The Bohr-Sommerfeld radiation criterion

$$\lim_{s \uparrow \infty} \int_{r=s} \mathrm{d}\mathbf{r}\, (\mathbf{r} \cdot \nabla - \mathrm{i}k)u = 0$$

One shows (see e.g.[3]) that these conditions admit a unique solution u. The *scattering amplitude* f is defined as

$$f(\mathbf{q}) := \lim_{r \uparrow +\infty} \mathrm{e}^{-\mathrm{i}kr} r u(r\mathbf{q}) \tag{2.1}$$

We define the capacity $C(\Omega)$ by

$$C(\Omega) = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\, \nu(\mathbf{p}) \quad \text{with } \nu \text{ the solution of} \quad \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\, \frac{\nu(\mathbf{p})}{|\mathbf{p} - \mathbf{r}|} = 1, \quad \mathbf{r} \in \partial\Omega \tag{2.2}$$

where $\mathrm{d}\sigma(\mathbf{p})$ is the measure on $\partial\Omega$, inherited from Lesbegue measure on $\mathbb{R}^3$. Our first result, Theorems 2.1 speaks about the low frequency regime.

Theorem 2.1. *Let σ and σ_T be as defined in* (1.3) *and* (1.4) *with $f(\mathbf{q})$ as defined in* (2.1)*. Let $C(\Omega)$ be the capacity as in* (2.2) *and $V(\Omega)$ the volume of a smooth compact body Ω, then*

$$\sigma_T \leq \sigma - \frac{4\pi}{3} k^2 C(\Omega) V(\Omega) + \mathcal{O}(k^4) \tag{2.3}$$

This follows by application of standard Green function techniques and an explicit computation. The next result, Theorem 2.2, is in the high-frequency regime. It can be easily deduced from earlier results, e.g. [11, 10], relying on the method of stationary phase.

Theorem 2.2. *Assume that the smooth, compact body Ω is strictly convex. There is $k_0 > 0$ such that for all $k > k_0$*

$$\sigma_T < \sigma \tag{2.4}$$

Remark 2.3. The relation between the scattering problem presented in Section 1 and the boundary value problem as presented above, is given as

$$\Psi(\mathbf{r}) = \mathrm{e}^{\mathrm{i}kz} + u(\mathbf{r}) \tag{2.5}$$

Remark 2.4. The condition that Ω is strictly convex, assures that f_{cl} exists. For example, if Ω is a cylinder with axis $\mathbf{e}$, then f_{cl} doesnot exist, nevertheless $R_{\mathrm{cl}}, \sigma_{\mathrm{cl}}$ can still be defined by (1.13), but now $R_{\mathrm{cl}} = 2\sigma_{\mathrm{cl}}$, which is the highest possible value for R_{cl}.

3 Proofs

3.1 Proof of Theorem 2.1

For bodies Ω with smooth boundary, one applies standard Green function techniques, see e.g. [9], to rewrite u, as defined in Section 2, in the form

$$u(\mathbf{r}) = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\, \mu(\mathbf{p}) \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{p}-\mathbf{r}|}}{|\mathbf{p}-\mathbf{r}|}, \quad \mathbf{r} \in \mathbb{R}^3 \setminus \Omega, \tag{3.1}$$

where μ is given as the jump in normal derivative of u on $\partial\Omega$,

$$\mu(\mathbf{p}) = - \lim_{\substack{\mathbf{r} \to \mathbf{p} \\ \mathbf{r} \in \mathbb{R}^3 \setminus \Omega}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{p}) + \lim_{\substack{\mathbf{r} \to \mathbf{p} \\ \mathbf{r} \in \Omega}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{p}) \tag{3.2}$$

where $\mathbf{n}$ is the outward normal at $\mathbf{p} \in \partial\Omega$ and $\frac{\partial}{\partial \mathbf{n}}$ stands for $\mathbf{n} \cdot \nabla$. The connection between the *scattering amplitude* f and μ is given by

$$f(\mathbf{q}) = \int \mathrm{d}\sigma(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}k\mathbf{p}\cdot\mathbf{q}} \mu(\mathbf{q}) \tag{3.3}$$

Our strategy will be to expand the functions $u(\mathbf{r})$ and $f(\mathbf{q})$ in powers of the wave number k and to investigate the behavior of $|f(\mathbf{q})|^2$ up to order k^3. The formal expansions in powers of k are justified by results in [3, 11] (in particular paragr. 2 Ch. 9 in [11]) assure that the expansions (3.4, 3.5) are convergent for all k.

We expand the function μ and f up to $\mathcal{O}(k^2)$,

$$\mu(\mathbf{p}) = \mu_0(\mathbf{p}) + ik\mu_1(\mathbf{p}) + (ik)^2\mu_2(\mathbf{p}) + \mathcal{O}(k^3), \quad \mathbf{p} \in \partial\Omega, \tag{3.4}$$

$$f(\mathbf{q}) = f_0(\mathbf{q}) + ikf_1(\mathbf{q}) + (ik)^2 f_2(\mathbf{q}) + \mathcal{O}(k^3), \quad \mathbf{q} \in S^2, \tag{3.5}$$

By using the boundary condition $u|_{\partial\Omega} = -\mathrm{e}^{\mathrm{i}kz}$ and (3.3), we have

$$\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \frac{\mu_0(\mathbf{p})}{|\mathbf{p}-\mathbf{r}|} = -1, \quad \mathbf{r} \in \partial\Omega, \tag{3.6}$$

$$\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \frac{\mu_1(\mathbf{p})}{|p-r|} \mathrm{d}\sigma(\mathbf{p}) + \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_0(\mathbf{p}) = -z, \quad \mathbf{r} \in \partial\Omega, \tag{3.7}$$

$$\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \frac{\mu_2(\mathbf{p})}{|p-r|} + \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_1(\mathbf{p}) + \frac{1}{2} \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_0(\mathbf{p}) |\mathbf{p} - \mathbf{r}| = -\frac{z^2}{2}, \quad \mathbf{r} \in \partial\Omega, \tag{3.8}$$

We evaluate the *scattering amplitude* f,

$$f_0(\mathbf{q}) = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_0(\mathbf{p}) \tag{3.9}$$

$$f_1(\mathbf{q}) = -\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_0(\mathbf{p})(\mathbf{p} \cdot \mathbf{q}) + \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_1(\mathbf{p}) \tag{3.10}$$

$$f_2(\mathbf{q}) = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_2(\mathbf{p}) - \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_1(\mathbf{p} \cdot \mathbf{q}) + \frac{1}{2} \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_0(\mathbf{p})(\mathbf{q} \cdot \mathbf{p})^2, \tag{3.11}$$

Let $\mathbf{R}_\mathrm{z}$ denote the inversion $z \to -z$, acting on subsets of $\mathbb{R}^3$. In particular

$$(\mathbf{x}, z) \in \mathbf{R}_\mathrm{z}\Omega \Leftrightarrow (\mathbf{x}, -z) \in \Omega \tag{3.12}$$

We split a function g on $\partial\Omega$ into 'symmetric' and 'antisymmetric' parts as follows

$$g^s(\mathbf{p}) = \frac{1}{2}[g(\mathbf{p}, \Omega) + g(\mathbf{R}_\mathrm{z}\mathbf{p}, \mathbf{R}_\mathrm{z}\Omega)] \qquad g^a(\mathbf{p}) = \frac{1}{2}[g(\mathbf{p}, \Omega) - g(\mathbf{R}_\mathrm{z}\mathbf{p}, \mathbf{R}_\mathrm{z}\Omega)], \qquad \mathbf{p} \in \partial\Omega \tag{3.13}$$

and similarly for functions h on S^2:

$$h^s(\mathbf{q}) = \frac{1}{2}[h(\mathbf{q}) + h(\mathbf{R}_\mathrm{z}\mathbf{q})] \qquad h^a(\mathbf{q}) = \frac{1}{2}[h(\mathbf{q}) - h(\mathbf{R}_\mathrm{z}\mathbf{q})], \qquad \mathbf{q} \in S^2 \tag{3.14}$$

With these definitions, we can immediately state:

$$\mu_0 = \mu_0^s \qquad C := -f_0 \text{ is constant} \tag{3.15}$$

$$\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \frac{\mu_1^a(\mathbf{p})}{|\mathbf{p} - \mathbf{r}|} = -z \tag{3.16}$$

$$\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \frac{\mu_1^s(\mathbf{p})}{|\mathbf{p} - \mathbf{r}|} = C \Rightarrow \mu_1^s = -C\mu_0 \tag{3.17}$$

$$\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \frac{\mu_2^a(\mathbf{p})}{|\mathbf{p} - \mathbf{r}|} = -\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_1^a(\mathbf{p}), \quad \mathbf{r} \in \partial\Omega \tag{3.18}$$

$$\begin{aligned} f_2^a(\mathbf{q}) &= -\cos\theta \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_1^s(\mathbf{p}) z(\mathbf{p}) = -\cos\theta \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_1^a(\mathbf{p}) z(\mathbf{p}) && (3.19) \\ &= \cos\theta C \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_0^s(\mathbf{p}) z(\mathbf{p}) - \cos\theta \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p}) \, \mu_1^a(\mathbf{p}) z(\mathbf{p}) && (3.20) \end{aligned}$$

Let u, v be harmonic functions on $\mathbb{R}^3 \setminus \Omega$, satisfying the boundary conditions

$$v|_{\partial\Omega} = -z \qquad u|_{\partial\Omega} = -1 \tag{3.21}$$

and apply Green's theorem

$$\int_R \mathrm{d}\mathbf{x}\,(u\triangle v - v\triangle u) = \int_{\partial R} \mathrm{d}\sigma(\mathbf{p})\,(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}) \tag{3.22}$$

with R being a smooth region in $\mathbb{R}^3 \setminus \Omega$, infinitesimally close to $\partial\Omega$ and extending far enough at infinity. The left-hand side of (3.22) vanishes, the right hand side gives

$$K(\Omega) := \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_1^a(\mathbf{p}) = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,z(\mathbf{p})\mu_0(\mathbf{p}) \tag{3.23}$$

For $\mathbf{q}, \mathbf{p} \in \mathbb{R}^3$, we write $z(\mathbf{p}), z(\mathbf{q})$ for their projections on the z-axis and $\mathbf{p}^\perp, \mathbf{q}^\perp$ for their projections on the $\mathbf{e}^\perp$-plane. Recall also that $\cos\theta = \mathbf{e}\cdot\mathbf{q}$. It follows that $\mathbf{p}\cdot\mathbf{q} = z(\mathbf{p})\cos\theta + \mathbf{q}^\perp\cdot\mathbf{p}^\perp$. Inserting (3.23) in (3.10) yields

$$\begin{aligned} f_1(\mathbf{q}) &= (1-\cos\theta)K + \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_1^s(\mathbf{p}) - \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,(\mathbf{q}^\perp\cdot\mathbf{p}^\perp)\mu_0(\mathbf{p}) && (3.24)\\ &= (1-\cos\theta)K + C^2 - \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,(\mathbf{q}^\perp\cdot\mathbf{p}^\perp)\mu_0(\mathbf{p}) && (3.25)\end{aligned}$$

Remark that by (3.18) and (3.23)

$$\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_2^a(\mathbf{p}) = -CK(\Omega) \tag{3.26}$$

We expand the scattering amplitude up to $\mathcal{O}(k^3)$;

$$|f(\mathbf{q})|^2 = f_0^2(\mathbf{q}) - 2k^2 f_0(\mathbf{q})f_2(\mathbf{q}) + k^2 f_1^2(\mathbf{q}) + \mathcal{O}(k^4) \tag{3.27}$$

and we use the above estimates to obtain

$$\begin{aligned} &\sigma - \sigma_T = \int_{S^2} \mathrm{d}\mathbf{q}\,\cos\theta|f(\mathbf{q})|^2 && (3.28)\\ =\;& 2k^2 \int_{S^2} \mathrm{d}\mathbf{q}\,\cos\theta\,(-f_0^s f_2^a + f_1^s f_1^a)\\ =\;& 2k^2 \int_{S^2} \mathrm{d}\mathbf{q}\,\cos\theta\Big\{C\Big(\cos\theta C\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_0^s(\mathbf{p})z(p) - \cos\theta\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_1^a(\mathbf{p})z(p)\Big)\\ -\;& \cos\theta K(K + C^2 - \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,(\mathbf{q}^\perp\cdot\mathbf{p}^\perp)\mu_0(\mathbf{p}))\Big\}\\ =\;& 2k^2 \int_{S^2} \mathrm{d}\mathbf{q}\,\cos^2\theta\Big(C^2K - C\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_1^a(\mathbf{p})z(p) - K^2 - KC^2 + K\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,(\mathbf{q}^\perp\cdot\mathbf{p}^\perp)\mu_0(\mathbf{p})\Big)\\ =\;& -\frac{4\pi}{3}k^2\Big(C\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_1^a(\mathbf{p})z(p) + K^2\Big)\end{aligned}$$

To obtain the last equality we used that

$$\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_0(\mathbf{p}) \int_{S^2} \mathrm{d}\mathbf{q}\, \cos^2\theta\,(\mathbf{q}^\perp \cdot \mathbf{p}^\perp) \tag{3.29}$$

vanishes since the second integrand is antisymmetric with respect to the transformation $(z(\mathbf{q}), \mathbf{q}^\perp) \to (z(\mathbf{q}), -\mathbf{q}^\perp)$. The rest of the proof will consist in showing that

$$-\frac{4\pi}{3}k^2 \left(C \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_1^a(\mathbf{p}) z(p) + K^2 \right) \geq \frac{4\pi}{3}k^2 CV \tag{3.30}$$

which immediately yields Theorem 2.1.

Let

$$v(\mathbf{r}) = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\frac{\mu_1^a(\mathbf{p})}{|\mathbf{p}-\mathbf{r}|}, \qquad v_{int} = v|_\Omega, \quad v_{ext} = v|_{\mathbb{R}^3\setminus\Omega} \tag{3.31}$$

Both v_{ext} and v_{int} are harmonic functions which can be continuously extended to $\partial\Omega$. We know that $v|_{\partial\Omega} = -z$ and hence necessarily $v_{int} = -z$. By Green function techniques (compare with (3.32)), we have

$$\mu_1^a(\mathbf{p}) = - \lim_{\substack{\mathbf{r}\to\mathbf{p} \\ \mathbf{r}\in\mathbb{R}^3\setminus\Omega}} \frac{\partial v_{ext}}{\partial \mathbf{n}}(\mathbf{p}) + \lim_{\substack{\mathbf{r}\to\mathbf{p} \\ \mathbf{r}\in\Omega}} \frac{\partial v_{int}}{\partial \mathbf{n}}(\mathbf{p}) \tag{3.32}$$

Calculate

$$\begin{aligned} -\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_1^a(\mathbf{p})z(\mathbf{p}) &= \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,v_{ext}(\mathbf{p}) \left(\frac{\partial v_{int}}{\partial \mathbf{n}}(\mathbf{p}) - \frac{\partial v_{ext}}{\partial \mathbf{n}}(\mathbf{p}) \right) &(3.33)\\ &= \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,z(\mathbf{p})\frac{\partial z(\mathbf{p})}{\partial \mathbf{n}} - \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,v_{ext}(\mathbf{p})\frac{\partial v_{ext}}{\partial \mathbf{n}}(\mathbf{p}) &(3.34)\\ &= V + \int_{\mathbb{R}^3\setminus\Omega} \mathrm{d}\mathbf{r}\,|\nabla v_{ext}(\mathbf{r})|^2, &(3.35) \end{aligned}$$

where V is the volume of Ω. To get the last equality, we applied the divergence theorem. Put

$$M(\Omega) = -\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,z(\mathbf{p})\mu_1^a(\mathbf{p}) - V = \int_{\mathbb{R}^3\setminus\Omega} \mathrm{d}\mathbf{r}\,|\nabla v_{ext}(\mathbf{r})|^2$$

Define also

$$u(\mathbf{r}) = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\frac{\mu_0(\mathbf{p})}{|\mathbf{p}-\mathbf{r}|}, \qquad u_{int} = u\big|_\Omega, \quad u_{ext} = u\big|_{\mathbb{R}^3\setminus\Omega} \tag{3.36}$$

Reasoning as above, we have that $u_{int} = -1$, hence

$$\begin{aligned} -C = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,\mu_1(\mathbf{p}) &= -\int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,u_{ext} \left(\frac{\partial u_{int}}{\partial \mathbf{n}}(\mathbf{p}) - \frac{\partial u_{ext}}{\partial \mathbf{n}}(\mathbf{p}) \right) &(3.37)\\ &= -\int_{\mathbb{R}^3\setminus\Omega} \mathrm{d}\mathbf{r}\,|\nabla u_{ext}(\mathbf{r})|^2, &(3.38) \end{aligned}$$

and

$$\begin{aligned} K = \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,z(\mathbf{p})\mu_1(\mathbf{p}) &= \int_{\partial\Omega} \mathrm{d}\sigma(\mathbf{p})\,v_{ext}(\mathbf{p})\frac{\partial u_{ext}}{\partial \mathbf{n}}(\mathbf{p}) &(3.39)\\ &= \int_{\mathbb{R}^3\setminus\Omega} \mathrm{d}\mathbf{r}\,(\nabla u_{ext}(\mathbf{r}) \cdot \nabla v_{ext}(\mathbf{r})), &(3.40) \end{aligned}$$

Since the functions $\nabla u_{ext}, \nabla v_{ext}$ are square integrable, the Cauchy-Schwarz inequality yields

$$K^2(\Omega) \le M(\Omega)C(\Omega) \tag{3.41}$$

which means that in (3.28), we can estimate

$$C(M+V) - K^2 \ge C(M+V) - CM = CV \tag{3.42}$$

which proves the inequality (3.30) since the LHS of (3.30) is $\frac{4\pi}{3}k^2\left(C(M+V)-K^2\right)$. This ends the proof.

3.2 Proof of Theorem 2.2

From techniques, based on the method of stationary phase, we know (see [7]) that for strictly convex bodies Ω,

$$|f(\mathbf{q})|^2 = |f_{\rm cl}(\mathbf{q})|^2 + O(1/k^2), \qquad S^2 \ni \mathbf{q} \neq \mathbf{e} \tag{3.43}$$

where the error estimate $O(1/k^2)$ is uniform in every compact subset of S^2 which does not contain $\mathbf{e}$. From [10], we know that

$$\lim_{k\to\infty} \int_{S^2} \mathrm{d}\mathbf{q}\, |f(\mathbf{q})|^2 = 2\sigma_{\rm cl} \tag{3.44}$$

Combining (3.43), (3.44) and (1.12), we get, in the sense of distribution on S^2,

$$\lim_{k\uparrow\infty} |f|^2 = |f_{\rm cl}|^2 + |\mathcal{I}|\delta_{\mathbf{e}} \tag{3.45}$$

where $\delta_{\mathbf{e}}$ is the Dirac delta distribution on S^2, peaked at $\mathbf{e} \in S^2$. An immediate consequence is

$$\lim_{k\to\infty} \sigma_T = R_{\rm cl}. \tag{3.46}$$

From the definition of σ_T and σ follows

$$\sigma_T \le 2\sigma, \qquad R_{\rm cl} \le 2\sigma_{\rm cl} \tag{3.47}$$

The second inequality is an equality only when the side of Ω, exposed to the incoming flow, is perpendicular to $\mathbf{e}$. Since we exclude this by assuming strict convexity, we get

$$R_{\rm cl} < 2\sigma_{\rm cl} \tag{3.48}$$

Let $\varepsilon = (2\sigma_{\rm cl} - R_{\rm cl})/2$. Using (3.45), we find a $k_0 > 0$ such that for $k > k_0$

$$\sigma_T < R_{\rm cl} + \varepsilon, \qquad \sigma > 2\sigma_{\rm cl} - \varepsilon, \tag{3.49}$$

and hence

$$\sigma_T < R_{\rm cl} + \varepsilon = 2\sigma_{\rm cl} - \varepsilon < \sigma \tag{3.50}$$

which ends the proof.

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